ECED 3300
Tutorial 7

Problem 1

A spherical capacitor has an inner radius a and the outer radius b and is filled with an inhomogeneous dielectric with $\epsilon(r) = \epsilon_0 k/r^2$. Show that the capacitance of the capacitor is given by

$$C = \frac{4\pi\epsilon_0 k}{b-a}.$$

Solution

Assume that the inner plate carries a charge $+Q$ and the outer one a charge $-Q$, respectively. Applying Gauss's law to a Gaussian sphere of a radius r, $a \leq r \leq b$, we obtain

$$\epsilon(r)E(r)4\pi r^2 = Q,$$

implying that

$$\frac{\epsilon_0 k}{r^2}4\pi r^2 E(r) = Q, \Longrightarrow E(r) = \frac{Q}{4\pi\epsilon_0 k}.$$

It then follows that the potential drop between the plates is

$$V = -\int d\mathbf{r}\cdot\mathbf{E} = -\int_b^a dr\frac{Q}{4\pi\epsilon_0 k} = \frac{Q(b-a)}{4\pi\epsilon_0 k}.$$

Finally, by definition,

$$C = \frac{Q}{V} = \frac{4\pi\epsilon_0 k}{b-a}.$$

Problem 2

Two point charges of the opposite signs, $Q_1 > 0$ and $Q_2 < 0$, are placed along the $z-$axis at the positions h_1 and h_2, respectively, above a grounded conducting xy-plane, as indicated in Fig.1.
(a) Find the **magnitude** *and* **direction** *of the force experienced by the charge Q_2.*
(b) Determine the electrostatic energy of the system.

Solution

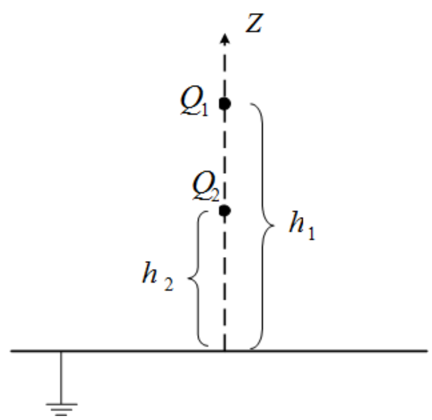

FIG. 1: Illustration to Problem 2

(a) The system of images consists of two charges, $-Q_1$ and $-Q_2$ placed at the positions $(0, 0, -h_1$ and $(0, 0, -h_2)$, respectively. The total force on charge Q_2 will then be,

$$\mathbf{F} = \mathbf{a}_z \frac{Q_1|Q_2|}{4\pi\epsilon_0(h_2 - h_1)^2} - \mathbf{a}_z \frac{Q_2^2}{4\pi\epsilon_0(2h_2)^2} + \mathbf{a}_z \frac{Q_1|Q_2|}{4\pi\epsilon_0(h_2 + h_1)^2}.$$

where we made use of Coulomb's law and the superposition principle.

(b) The energy of the system corresponds to the energy of the system of images with the caveat that the potential below the grounded plane is always zero. Therefore,

$$W_E = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2 - Q_1 \times 0 - Q_2 \times 0)$$

Using the superposition principle we can obtain the potential at the position of Q_1 due to the other charges:

$$V_1 = \frac{Q_2}{4\pi\epsilon_0(h_1 - h_2)} - \frac{Q_1}{8\pi\epsilon_0 h_1} - \frac{Q_2}{4\pi\epsilon_0(h_1 + h_2)}.$$

By the same token,

$$V_2 = \frac{Q_1}{4\pi\epsilon_0(h_1 - h_2)} - \frac{Q_2}{8\pi\epsilon_0 h_2} - \frac{Q_1}{4\pi\epsilon_0(h_1 + h_2)}.$$

Thus,

$$W_E = \frac{Q_1 Q_2}{4\pi\epsilon_0(h_1 - h_2)} - \frac{Q_1^2}{16\pi\epsilon_0 h_1} - \frac{Q_2^2}{16\pi\epsilon_0 h_2} - \frac{Q_1 Q_2}{4\pi\epsilon_0(h_1 + h_2)}.$$

Problem 3

Two capacitor of capacitances C_1 and C_2 are connected in series and are connected to the source of voltage V. Determine the voltage drops across each capacitor.

Solution

Assume the sought voltage drops are V_1 and V_2. Because the capacitors are connected in series, they carry identical charges such that $C_1V_1 = C_2V_2$. Further, the voltage drops satisfy the equation $V_1 + V_2 = V$. Solving these two equations simultaneously give the answer

$$V_1 = \frac{C_2 V}{C_1 + C_2}, \qquad V_2 = \frac{C_1 V}{C_1 + C_2}.$$